

# SYNCHRONIZATION BY PASSAGE THROUGH RESONANCES IN NON-NEUTRAL PLASMAS

*Lazar Friedland*

*Hebrew University of Jerusalem and UC Berkeley (on sabbatical)*

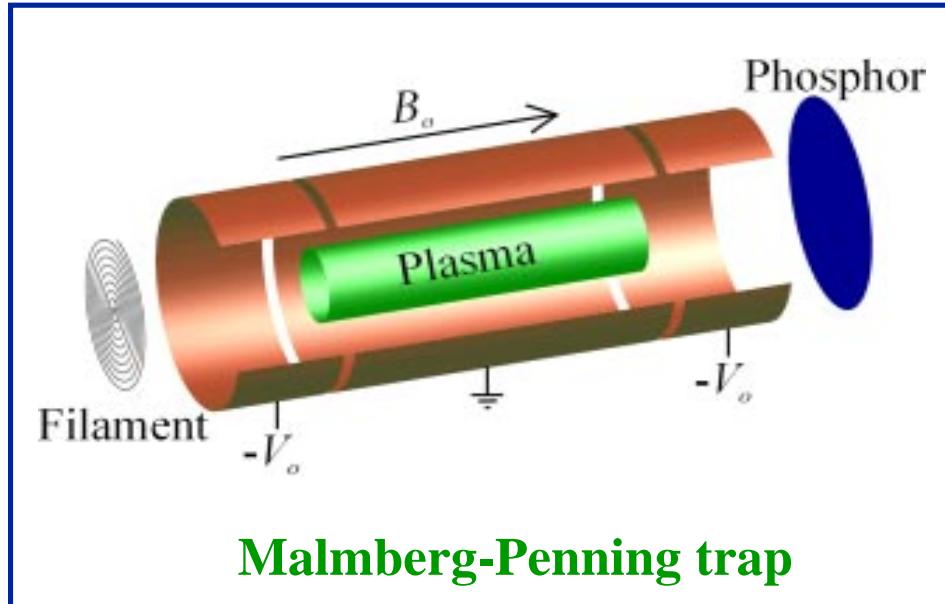
## OUTLINE

- A. Magnetized electron clouds and uniform V-states**
- B. Pattern formation by synchronization**
- C. Analytic approach and threshold for synchronization**
- D. Excitation of nonuniform V-States and stability**
- E. Subharmonic synchronization**
- F. Summary**

Collaborators:

Arkadi Shagalov (Ekaterinburg) , Joel Fajans (UC Berkeley)

## A. MAGNETIZED ELECTRON CLOUDS AND V-STATES



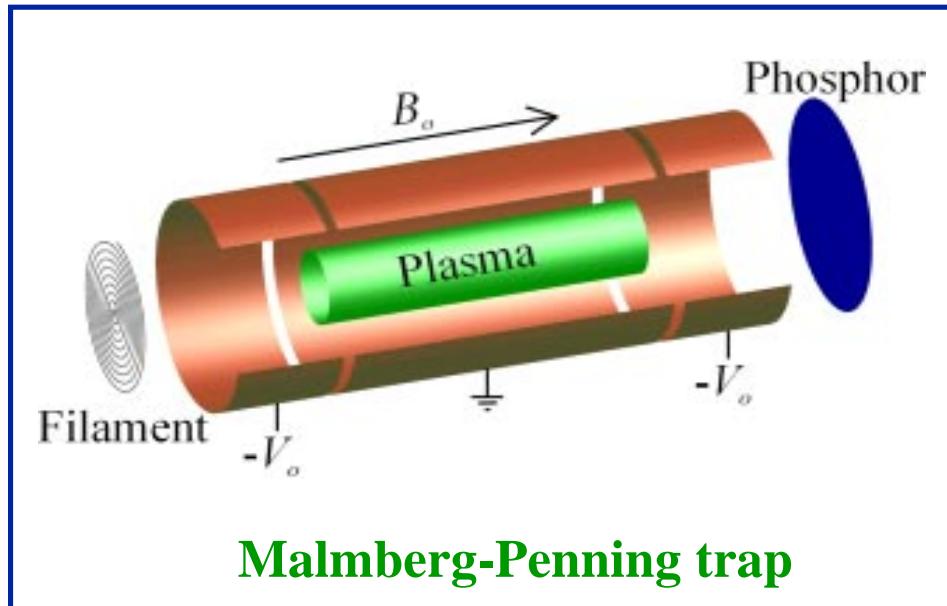
$$\overset{\text{r}}{V}/c = -\nabla\phi \times \hat{\mathbf{e}}_z / B_0$$

Drift-Poisson system

$$\partial n / \partial t + \overset{\text{r}}{V} \cdot \nabla n = 0$$

$$\Delta\phi = 4\pi en$$

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### 2D IDEAL FLUIDS

$$\overset{r}{V} = -\nabla\psi \times \hat{\mathbf{e}}_z$$

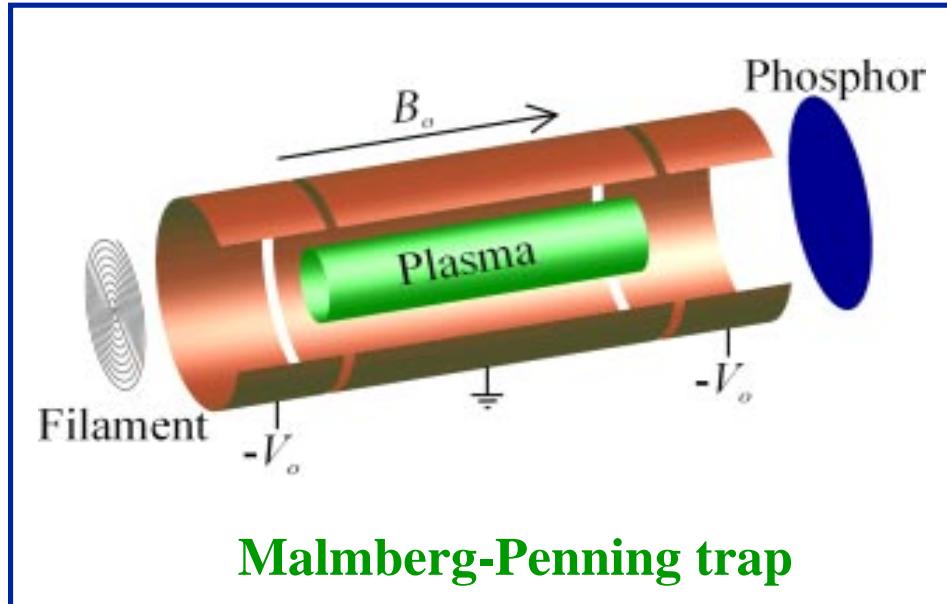
$$\overset{r}{\omega} = \nabla \times \overset{r}{V} = \Delta\psi \hat{\mathbf{e}}_z$$

### Euler's equations

$$\partial\omega / \partial t + \overset{r}{V} \cdot \nabla\omega = 0$$

$$\Delta\psi = \omega$$

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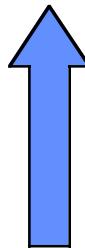
$$\partial\omega / \partial t + \overset{r}{V} \cdot \nabla\omega = 0$$

$$\Delta\psi = \omega$$

Analog:

$$\omega = nB_0 / (4\pi ec)$$

$$\psi = (B_0 / c)\phi$$

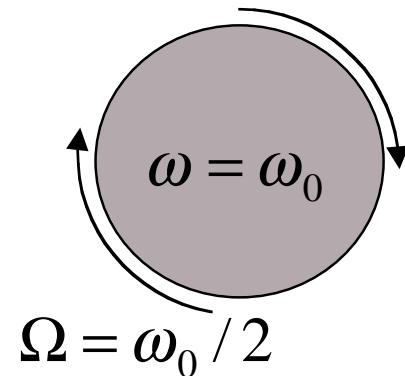


Euler's equations

$$\partial \boldsymbol{\omega} / \partial t + \mathbf{V} \cdot \nabla \boldsymbol{\omega} = 0$$

$$\Delta \psi = \boldsymbol{\omega}$$

Simplest Solution



$$\Omega = \omega_0 / 2$$

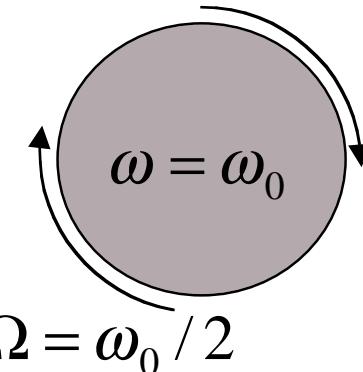
Solid body rotation

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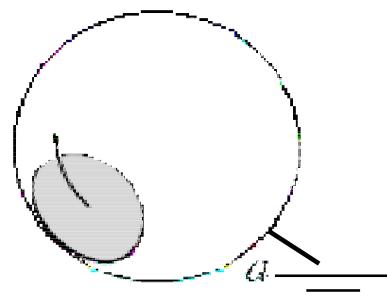


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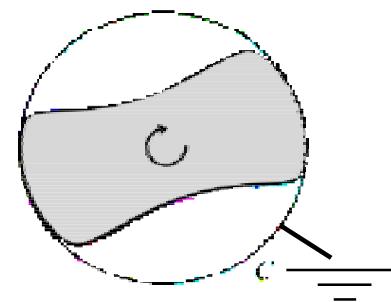
## Uniform V-STATES

**Q:** Are there noncircular uniformly rotating constant density plasma equilibria ?

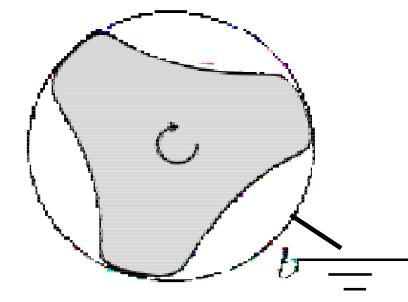
**A:** Yes, the V-states (Deem, Zabusky, 1979)



$m=1$  diocotron mode



2-fold V-state



3-fold V-state

How to create a nontrivial V-state by a small external perturbation ?

SOLUTION: Passage through resonance and synchronization

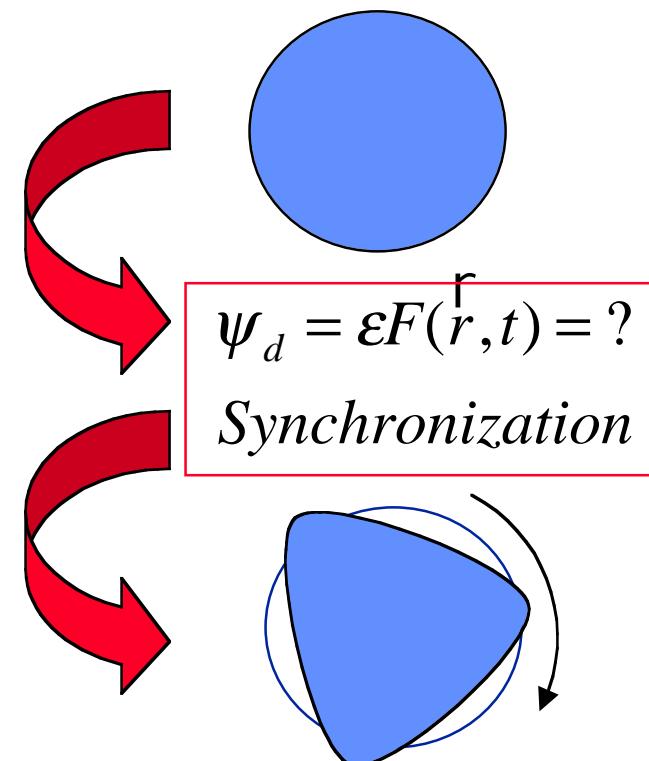
**IN:** Simple circular plasma column



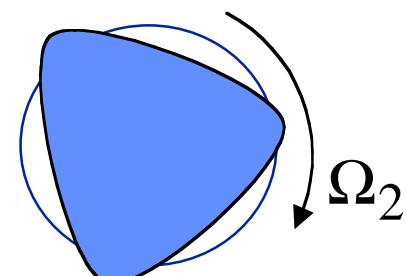
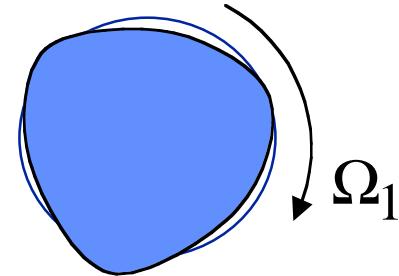
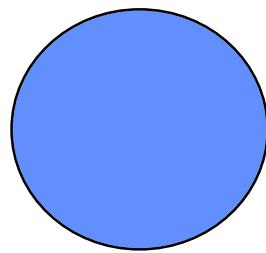
**ADD:** chirped frequency perturbation



**OUT:** desired V-state



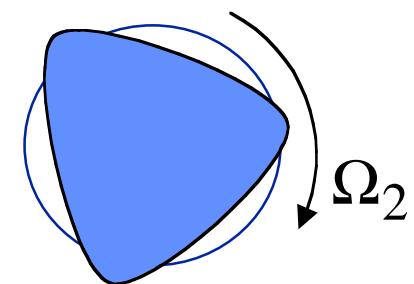
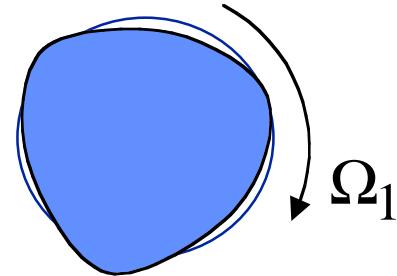
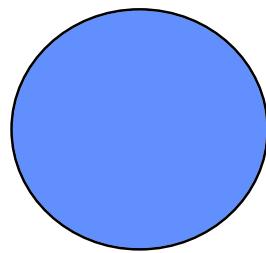
View uniform V-states as a continuous family of  
area preserving deformations of a circle



Rotation frequency is a function of the amplitude of deformation

$$\Omega = \Omega(a)$$

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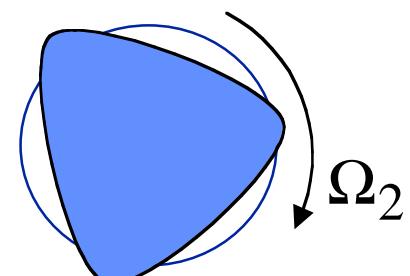
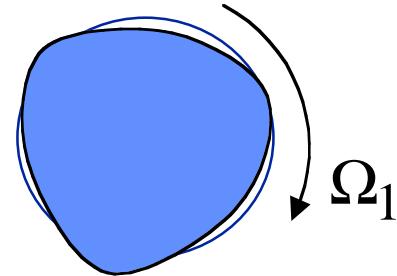
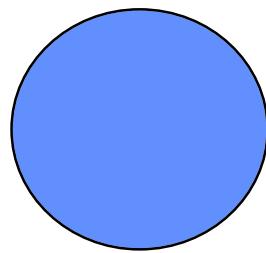
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time-evolving family of V-states  $\leftarrow a = a(t)$

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$\omega = const$

$Area = const$

numerical approach

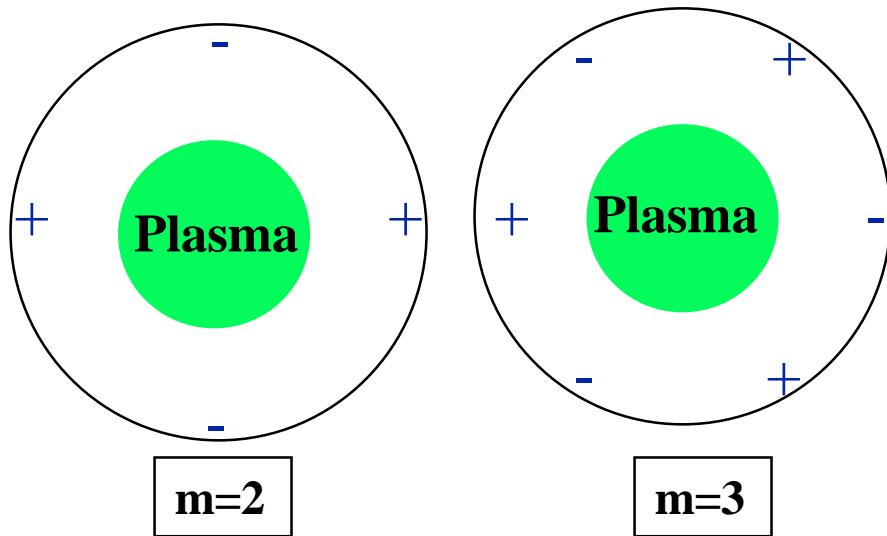
CONTOUR DYNAMICS

## Realization of V-states by synchronization

### SYNCHRONIZING POTENTIAL (passing through linear resonance)

$$\psi_d = \varepsilon \cos[\int \omega_d(t)dt] r^m \cos(m\phi)$$

$$\omega_d(t) = \Omega_0 - \alpha t$$

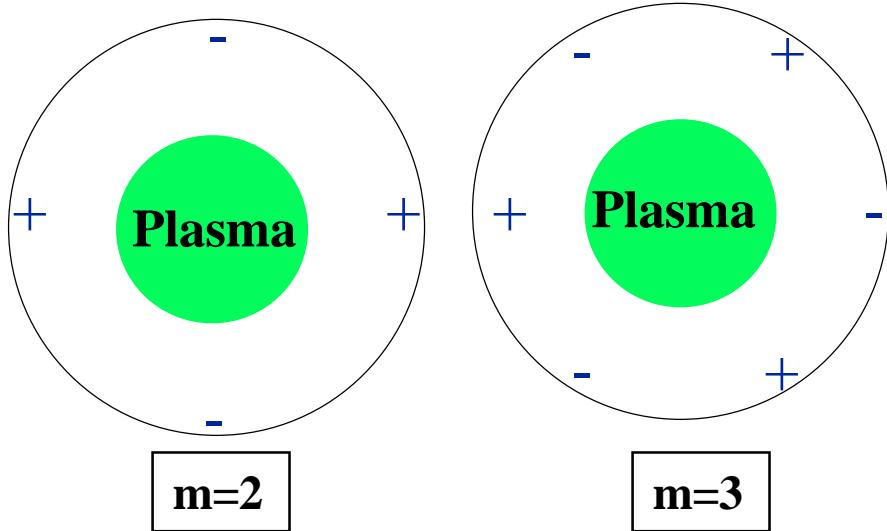


## Realization of V-states by synchronization

14

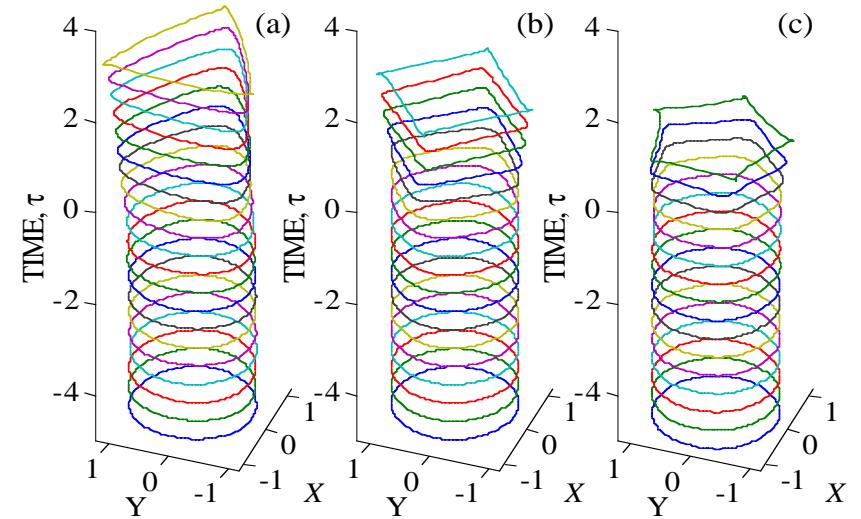
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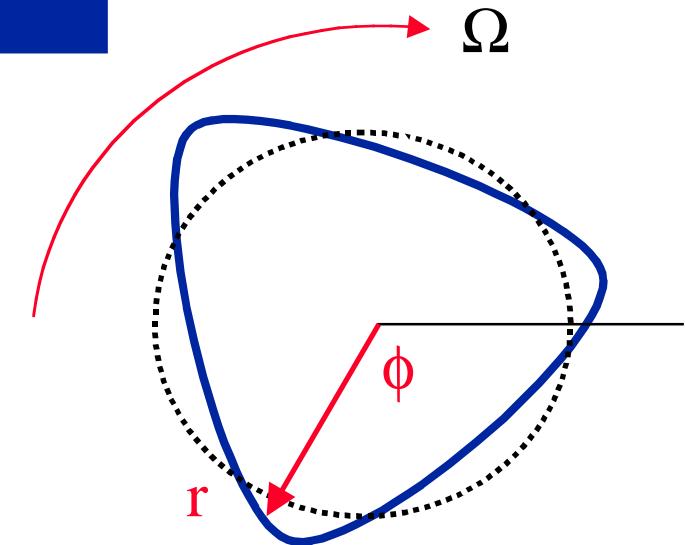
### V-states in unbounded plasma

CD simulations



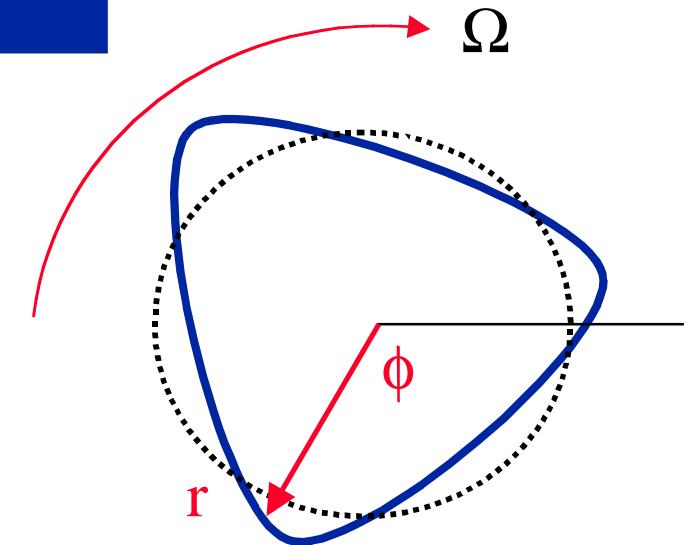
## C. THRESHOLD FOR SYNCHRONIZATION (ANALYTIC APPROACH)

Unperturbed V-states :  
Nonlinear traveling waves on a circle



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Unperturbed V-states :  
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Weakly nonlinear solution: (Su, 1979)

$$r \approx a_0 + a_1 \cos[m(\phi - \Omega t)] + a_2 \cos[2m(\phi - \Omega t)]$$

$$a_2 \sim a_1^2 \quad \Omega \approx \Omega_0 - \frac{\omega(m-1)}{4} a_1^2$$

Linear frequency:  $\Omega_0 = \frac{\omega(m-1)}{2m}$

## RESONANTLY DRIVEN V-STATES

$$r(\phi, t) \approx a_2(t) + a_1(t) \cos\{m[\phi - \theta(t)]\} + a_2(t) \cos\{2m[\phi - \theta(t)]\}.$$

$a_0(t), a_1(t), a_2(t)$

phase mismatch  $\Phi(t) = \int \omega_d(t) dt - \theta$

Slow functions  
of time

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Slow evolution equations



$$\begin{aligned} da_1/dt &= -\frac{\epsilon m}{2} \sin m\Phi \\ d\Phi/dt &= \beta a_1^2 - \alpha t - \frac{\epsilon}{2a_1} \cos m\Phi \end{aligned}$$

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### RENORMALIZED PROBLEM

$$\begin{aligned} \tau &= (m\alpha)^{1/2} t \\ \mu &= (\varepsilon/2)(m/\alpha)^{3/4} \\ \Psi &= [\beta^{1/2}(m/\alpha)^{1/4} a_1] \exp(im\Phi) \end{aligned}$$

### GENERIC NLS-TYPE EQUATION

$$i \frac{d\Psi}{d\tau} + (|\Psi|^2 - \tau)\Psi = \mu$$

### SINGLE PARAMETER SYSTEM

## THRESHOLD PHENOMENON

$$i \frac{d\Psi}{d\tau} + (|\Psi|^2 - \tau)\Psi = \mu$$

Two asymptotic solutions at  $\tau \rightarrow +\infty$  subject to  $\Psi|_{\tau \rightarrow -\infty} = 0$

Saturated solution

$$\Psi = a_1 \exp(-i\tau^2/2)$$

$$a_1 = \text{const}; \quad \Phi \sim \tau^2/2$$

Phase locked solution

$$\Psi = \tau^{1/2}$$

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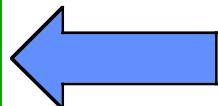
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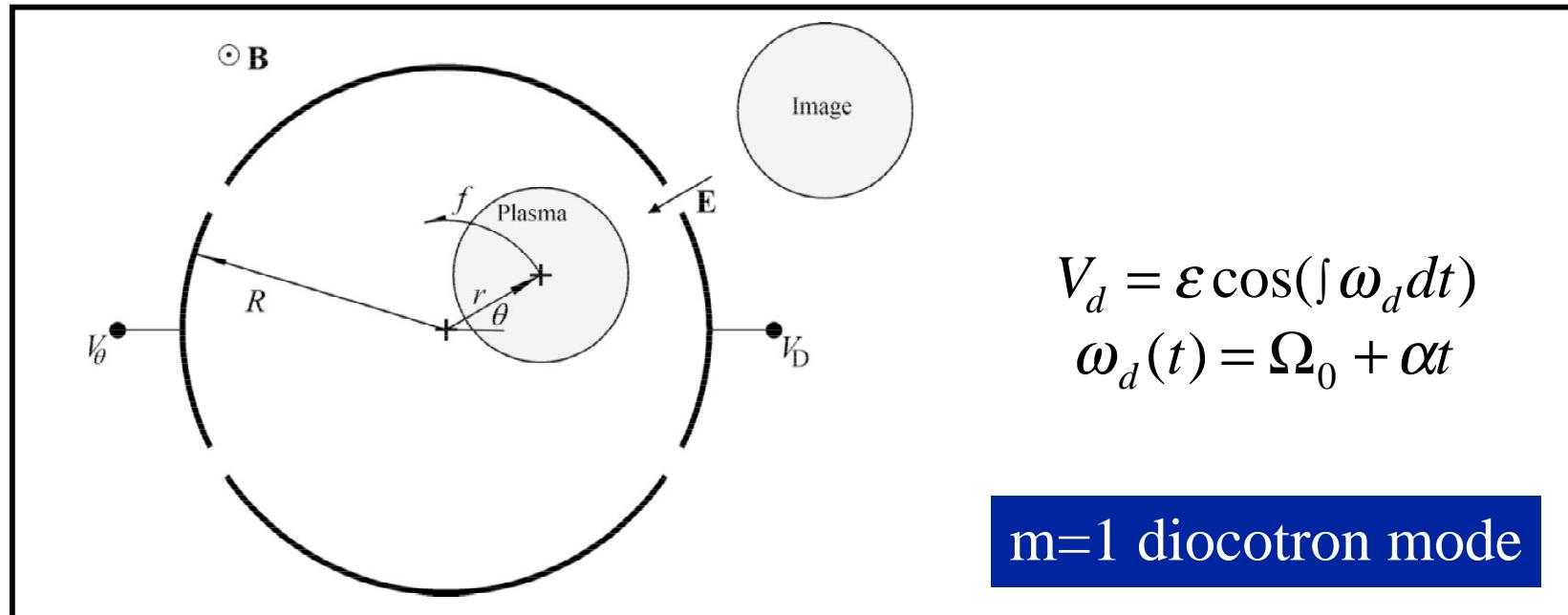
**SINGLE PARAMETER  $\mu$  CONTROLS THE BIFURCATION**

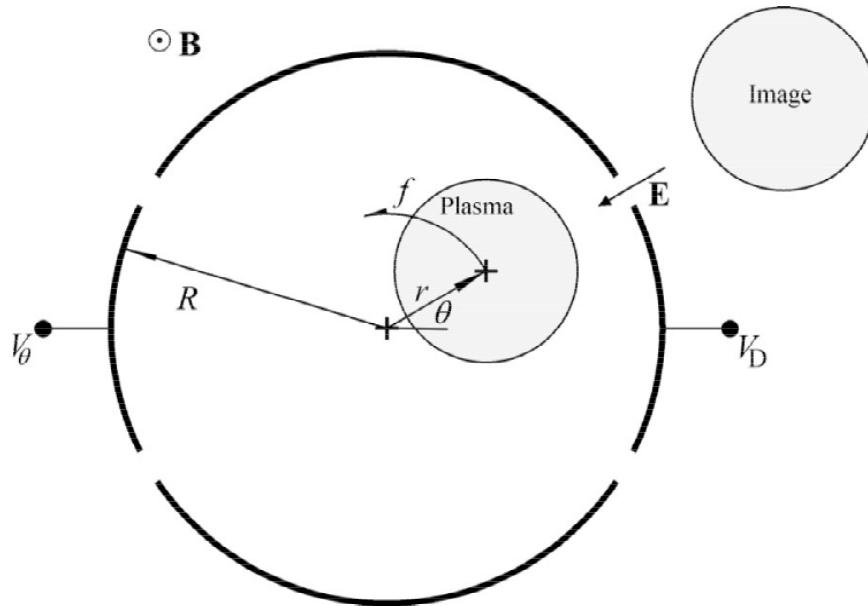
Phase locked solution for  
 $\mu > \mu_{th} = 0.411$

$$\varepsilon > \varepsilon_{th} \sim \beta^{-1/2} \alpha^{3/4}$$

First observed in experiments  
on magnetized electron clouds  
*Fajans, Gilson, Friedland*  
*PRL 82, 4444 (1999)*

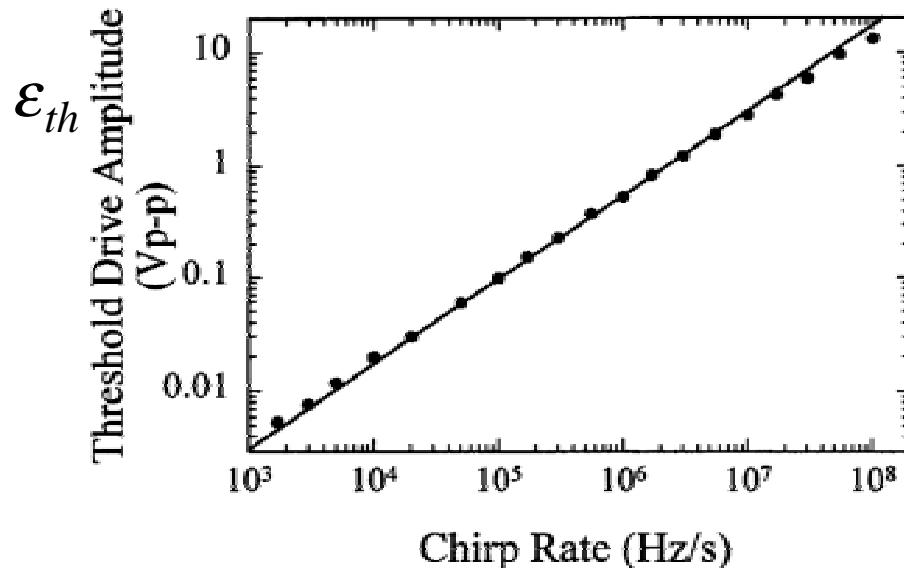






$$V_d = \varepsilon \cos(\int \omega_d dt)$$
$$\omega_d(t) = \Omega_0 + \alpha t$$

$m=1$  diocotron mode



$$\varepsilon > \varepsilon_{th} \sim \beta^{-1/2} \alpha^{3/4}$$

## FOR NON-BELIEVERS

(Animation by J. Fajans)

$$\alpha = 0.001$$

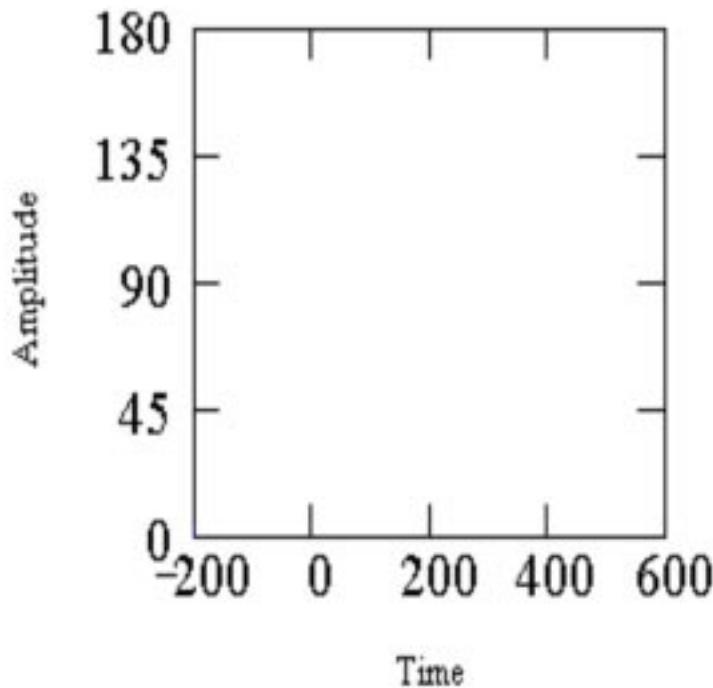


Time = -200

DriveFrequency = 1.20

$$\varepsilon = .03$$

### Autoresonantly Driven Pendulum

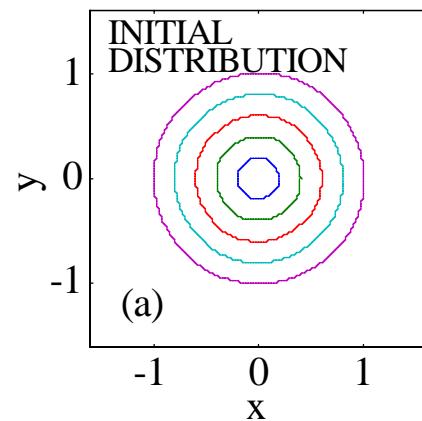


$$\varepsilon_{\text{crit}} = 0.02$$

$$\varepsilon = 0.01$$

AGAIN: PATTERN FORMATION BY PASSAGE THROUGH RESONANCE

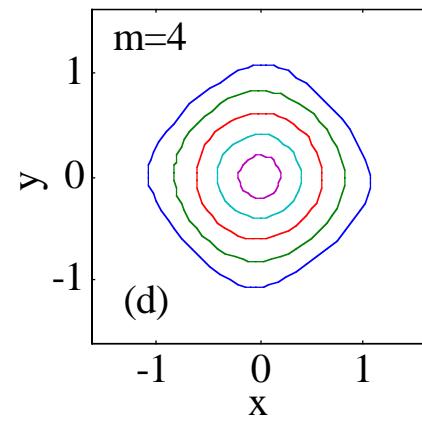
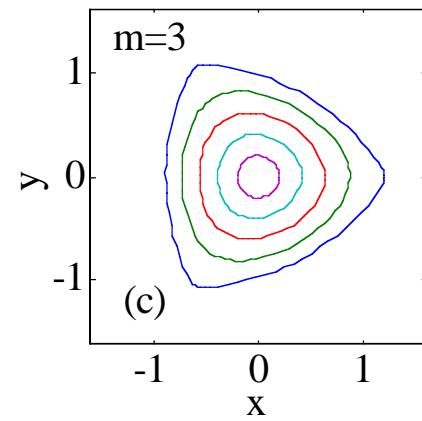
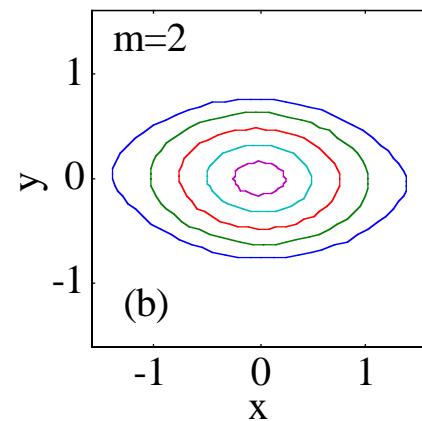
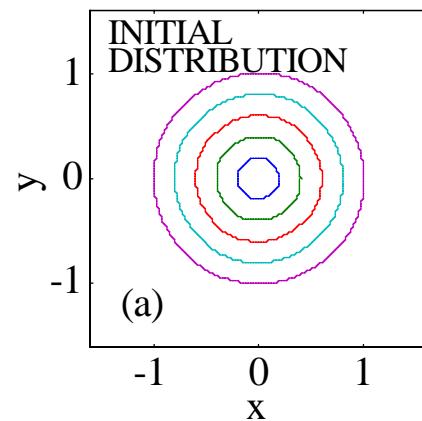
$$\omega(r) = \omega_0(1 - kr^2)$$



AGAIN: PATTERN FORMATION BY PASSAGE THROUGH RESONANCE

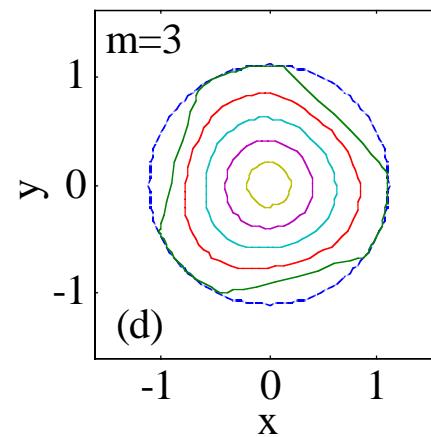
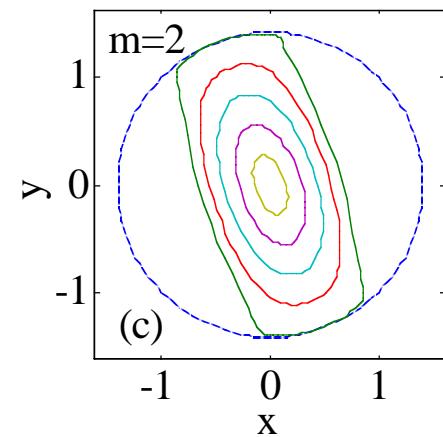
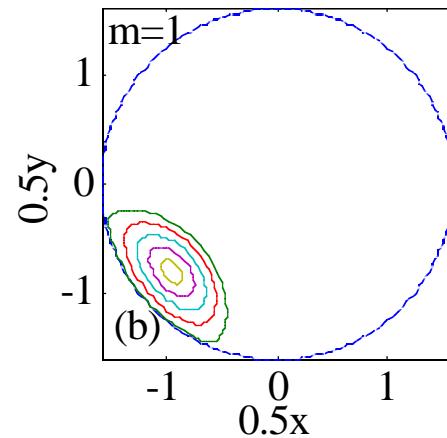
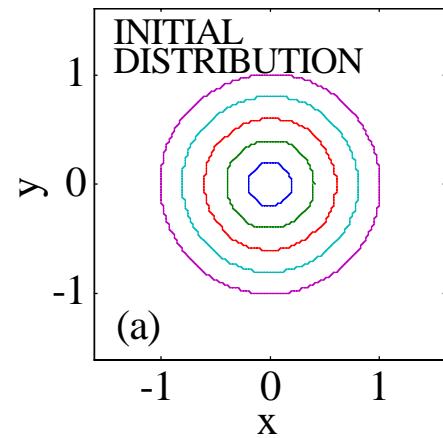
$$\omega(r) = \omega_0(1 - kr^2)$$


Muticontour  
CD simulations



## NONUNIFORM V-STATES IN BOUNDED SPACE

**m=1 Diocotron mode  
Berkeley experiments**



## MODAL STRUCTURE OF NONUNIFORM V-STATES

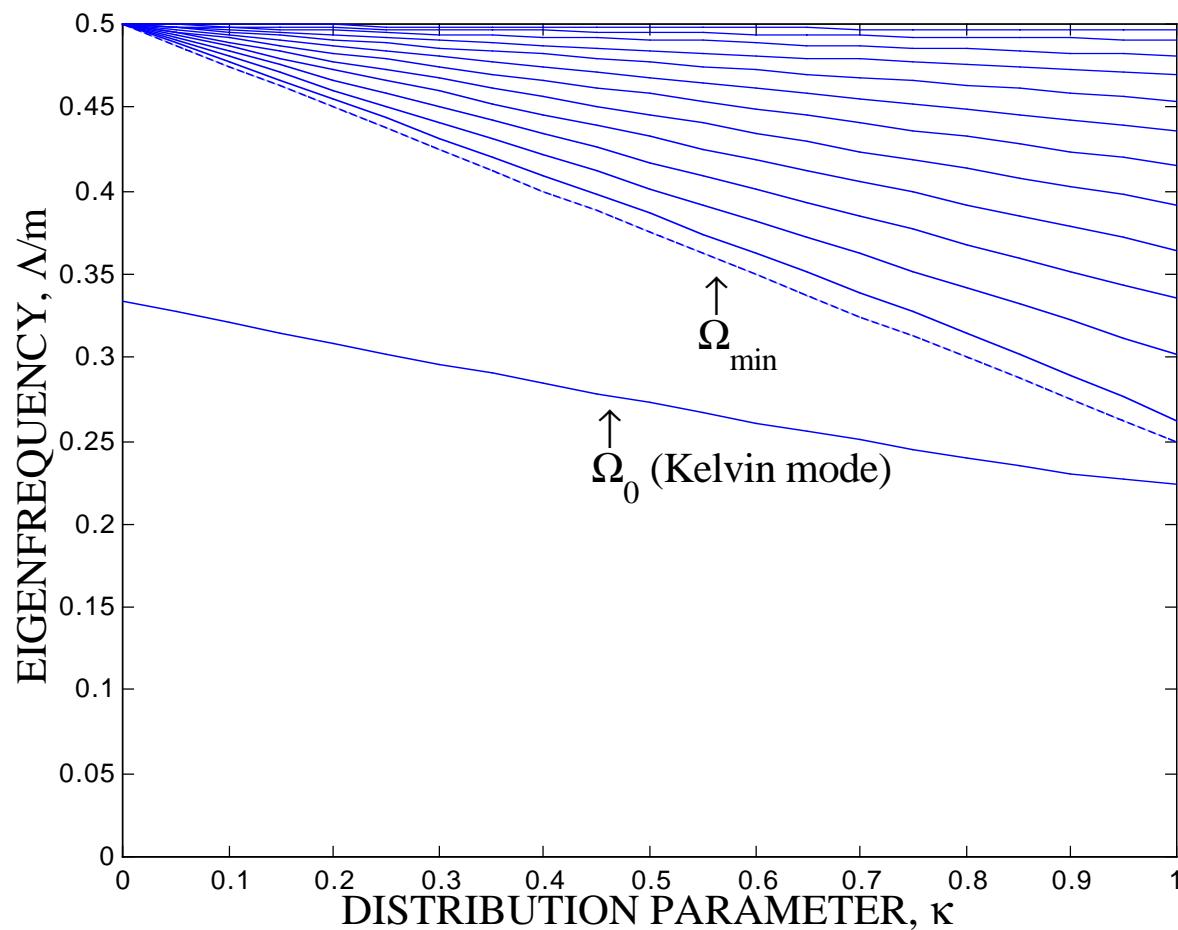
Initial density distribution  
 $\omega(r) = \omega_0(1 - kr^2)$

Superposition of N density slices  
N linear, m-fold symmetric modes

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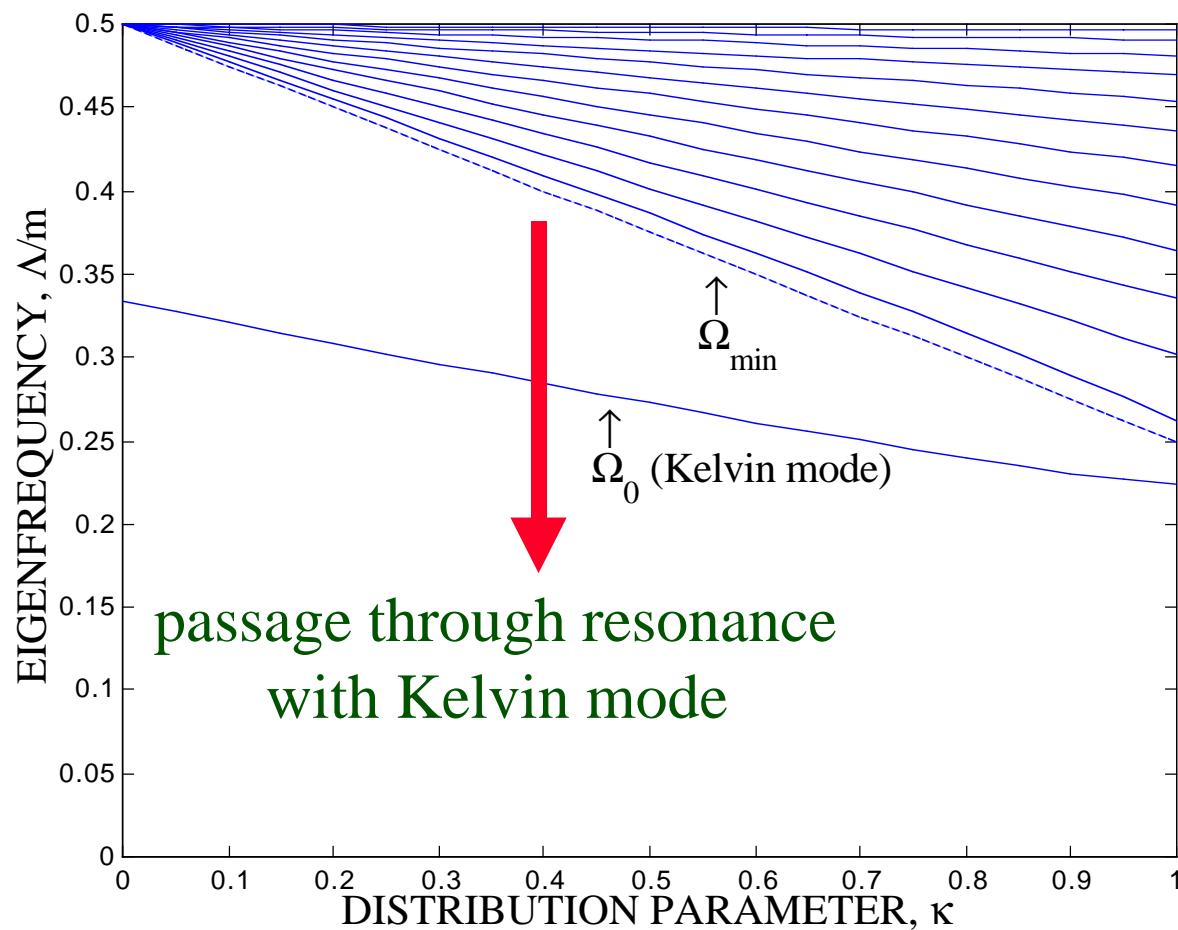
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Superposition of  $N$  density slices  
 $N$  linear,  $m$ -fold symmetric modes



## Weakly nonlinear theory of synchronized vortex pack

Each vorticity slice

$$a_n = a e_n, \quad n = 1, \dots, N, \quad |\mathbf{e}|^2 = 1$$

$$\frac{da}{dt} = -\frac{\varepsilon m}{2} \sin m\Phi$$

$$\frac{d\Phi}{dt} = \beta a^2 - \alpha t - \frac{\varepsilon}{2a} \cos m\Phi$$

Again, nonlinear Schrodinger-type equation for  $\Psi = a \exp(i\Phi)$

$$i \frac{d\Psi}{d\tau} + (|\Psi|^2 - \tau)\Psi = \mu$$

Full description of resonant interaction:

Threshold for resonant capture  $\varepsilon_{th} \sim \beta^{-1/2} \alpha^{3/4}$

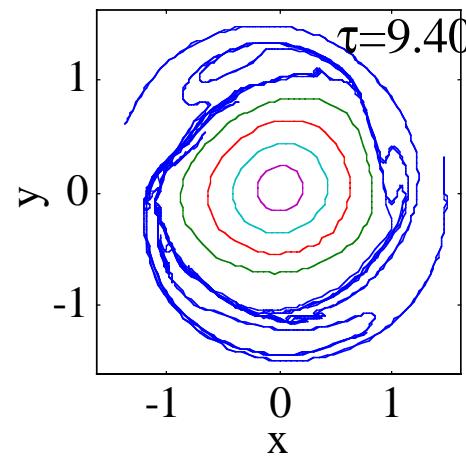
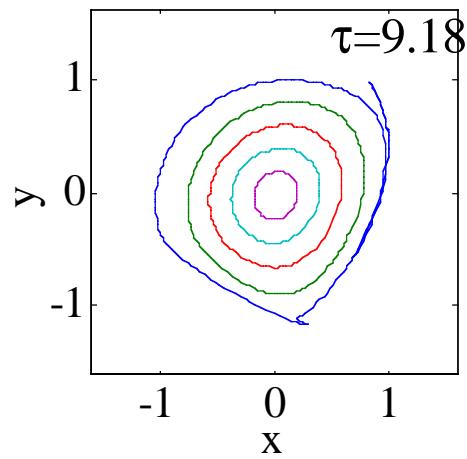
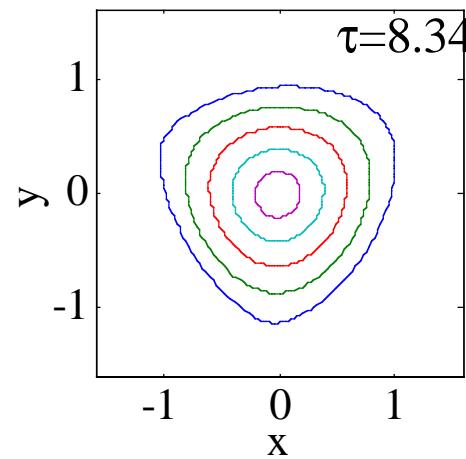
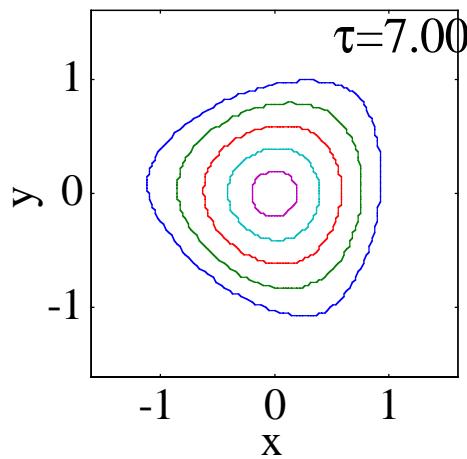
Continuing phase locking  $\rightarrow$  pattern formation

# STABILITY OF NONUNIFORM V-STATES ?

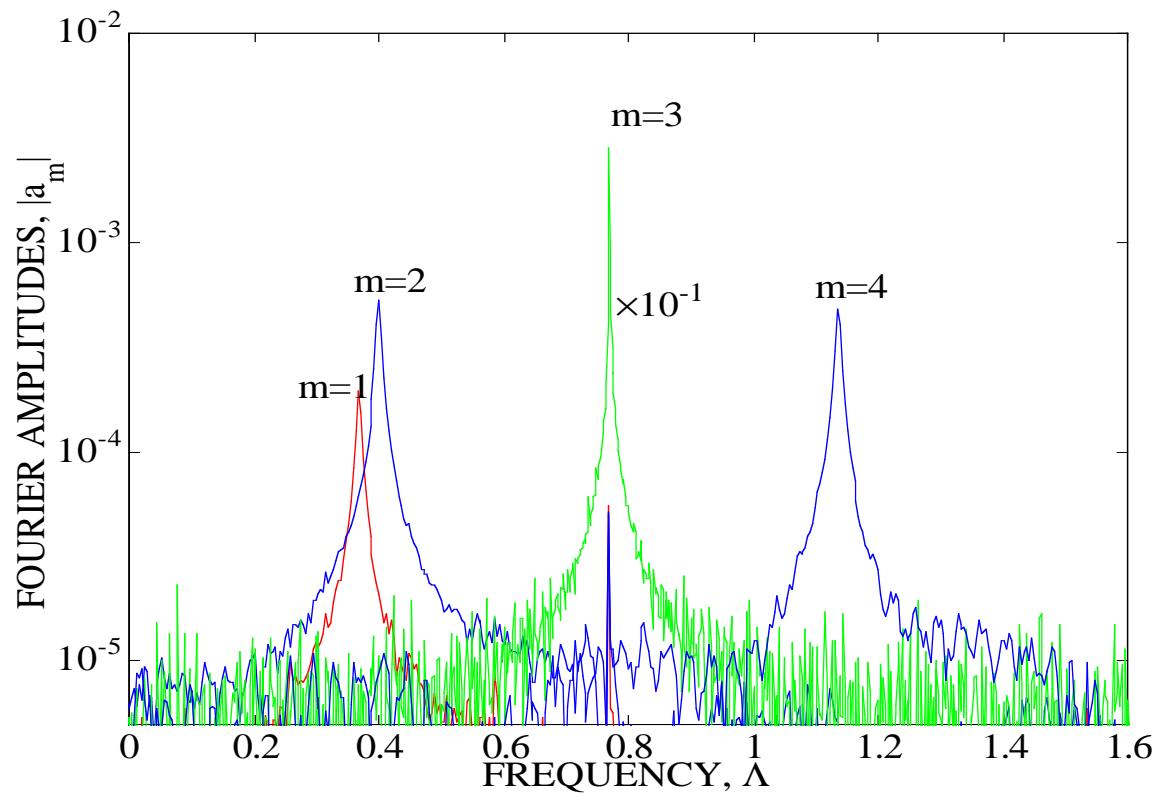
$m=2$  V-state is stable  
 $m=3,4,\dots$  are not

4

Instability of  $m=3$  state



## Numeical Spectrum of Developing Instability



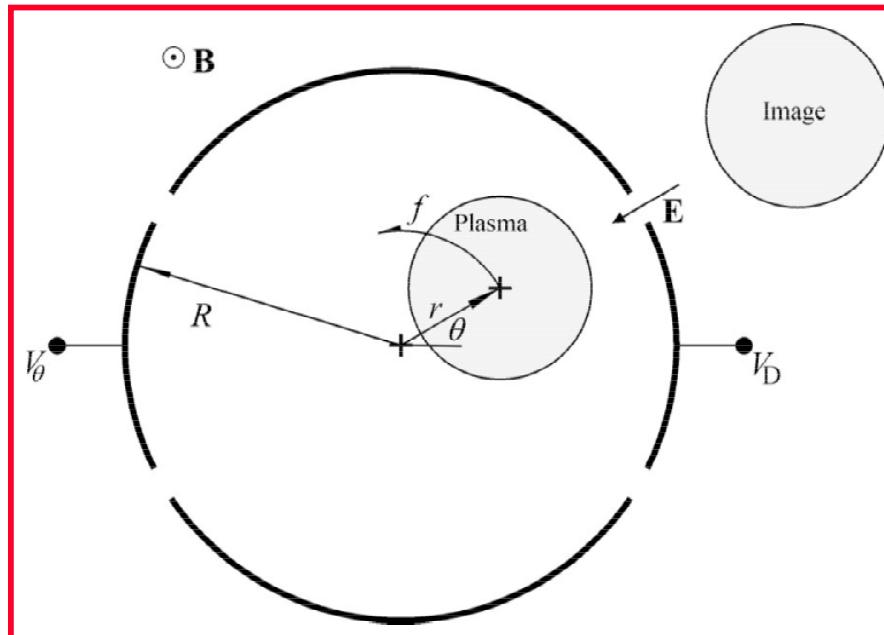
$$\Lambda_3 = \Lambda_1 + \Lambda_2$$

$$\Lambda_4 = \Lambda_1 + \Lambda_3$$

Resonant 3-wave interactions for  $m=1,2,3$   $m=1,3,4$   
 $m=1,2,4$  modes grow exponentially

## E. SUBHARMONIC SYNCHRONIZATION

2



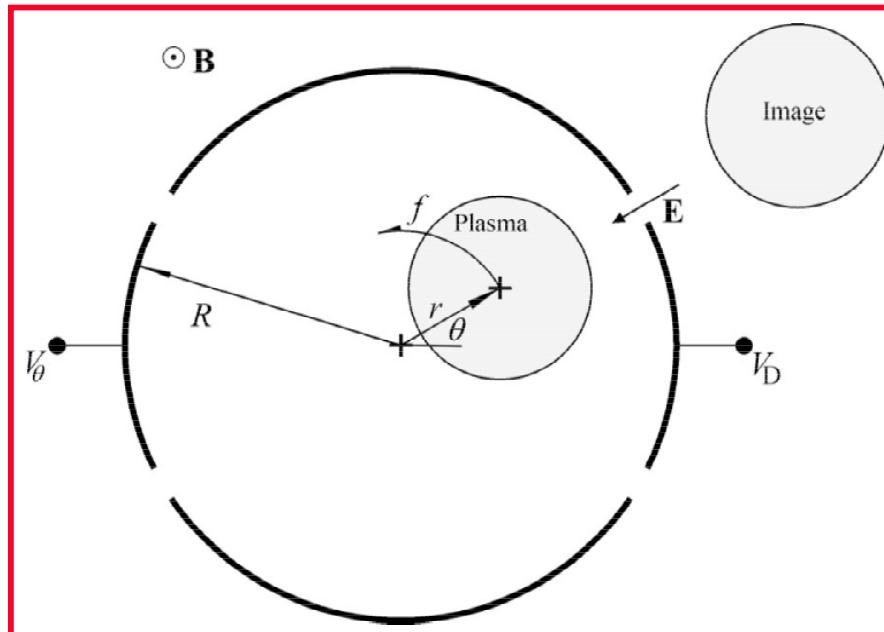
Can we excite  $m=1$  diocotron mode by passage through subharmonic frequencies ???

$$V_d = \varepsilon \cos(\omega_d t)$$

$$\omega_d(t) = (\Omega_0 + \alpha t)/n$$

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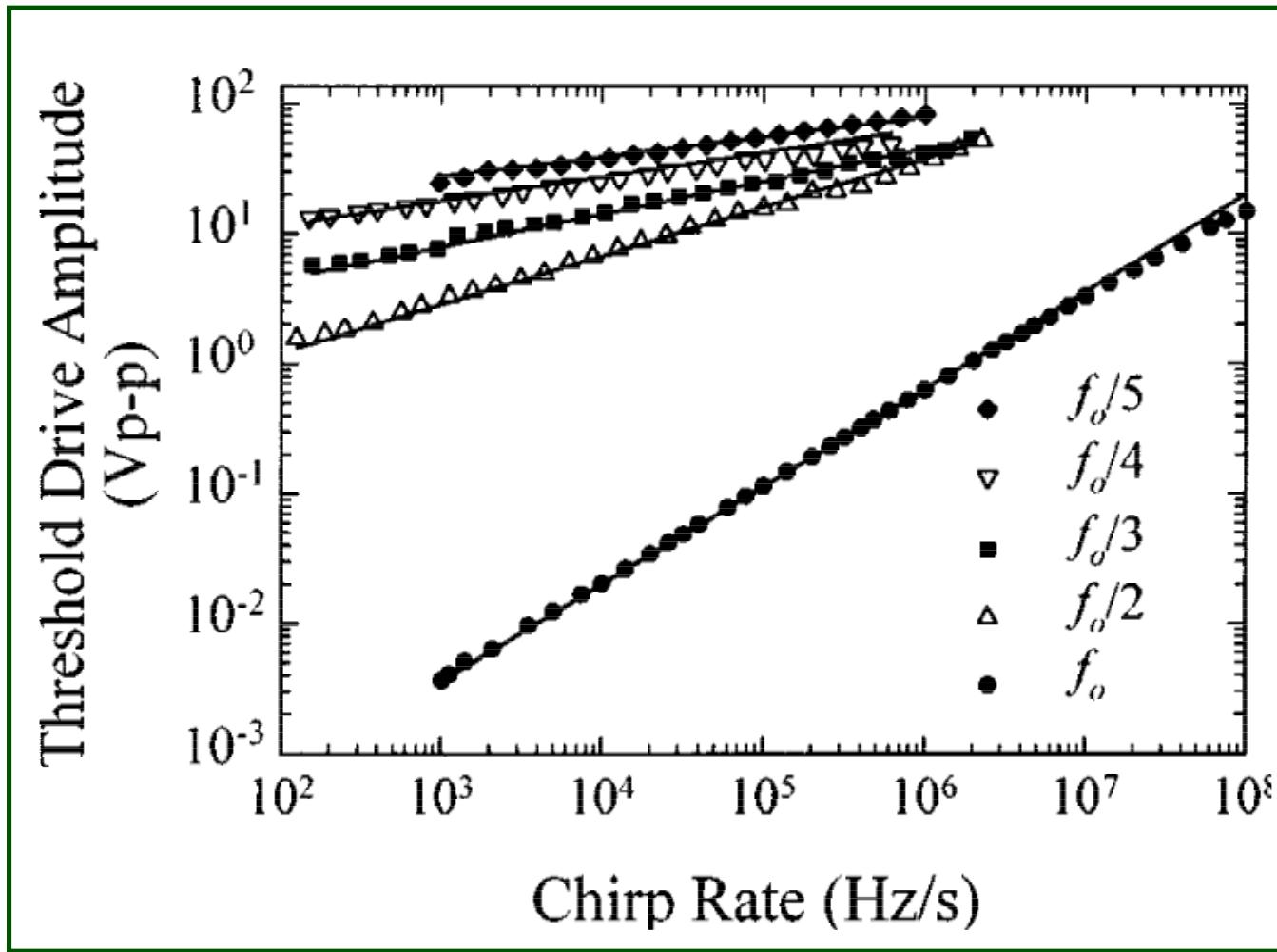
$$V_d = \varepsilon \cos(\omega_d t)$$

$$\omega_d(t) = (\Omega_0 + \alpha t)/n$$

- Drive amplitude  $\varepsilon$  at  $\omega_d(t) = (\Omega_0 + \alpha t)/n$
- Nonresonant response  $a \sim \varepsilon$  at  $\omega_d(t) = (\Omega_0 + \alpha t)/n$
- $n$ -th order nonlinear response  $a_n \sim \varepsilon^n$  at  $\omega'_d(t) = n\omega_d(t) = \Omega_0 + \alpha t$
- Effective resonant drive of amplitude  $\varepsilon' \sim \varepsilon^n$
- Usual threshold  $\varepsilon'_{th} \sim \beta^{-1/2} \alpha^{3/4}$   $\longrightarrow$   $\varepsilon_{th} \sim \beta^{-1/(2n)} \alpha^{3/(4n)}$

## Experimental confirmation

$$\varepsilon > \varepsilon_{th} \sim \beta^{-1/(2n)} \alpha^{3/(4n)}$$



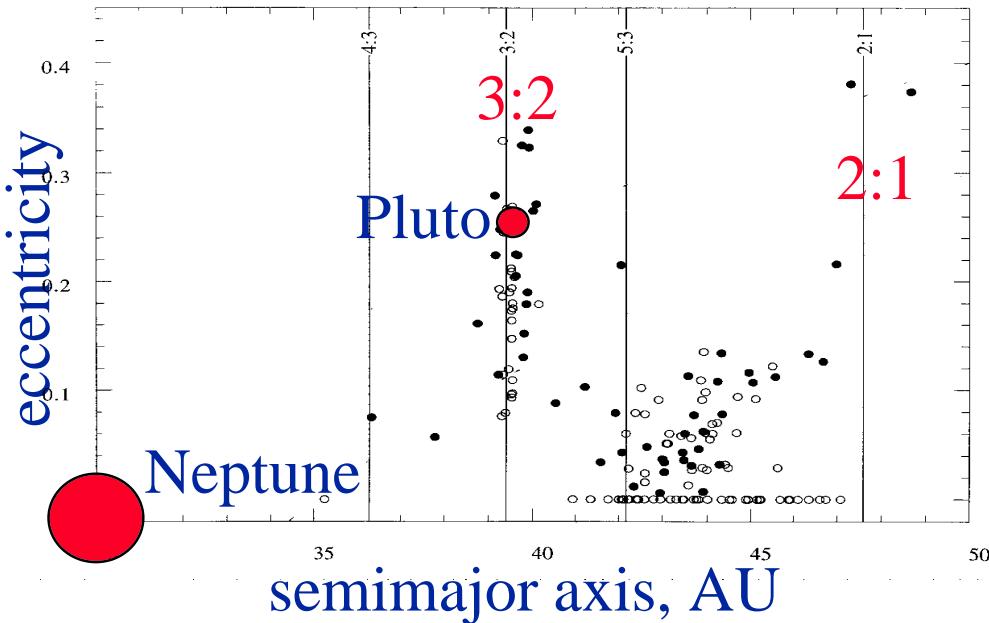
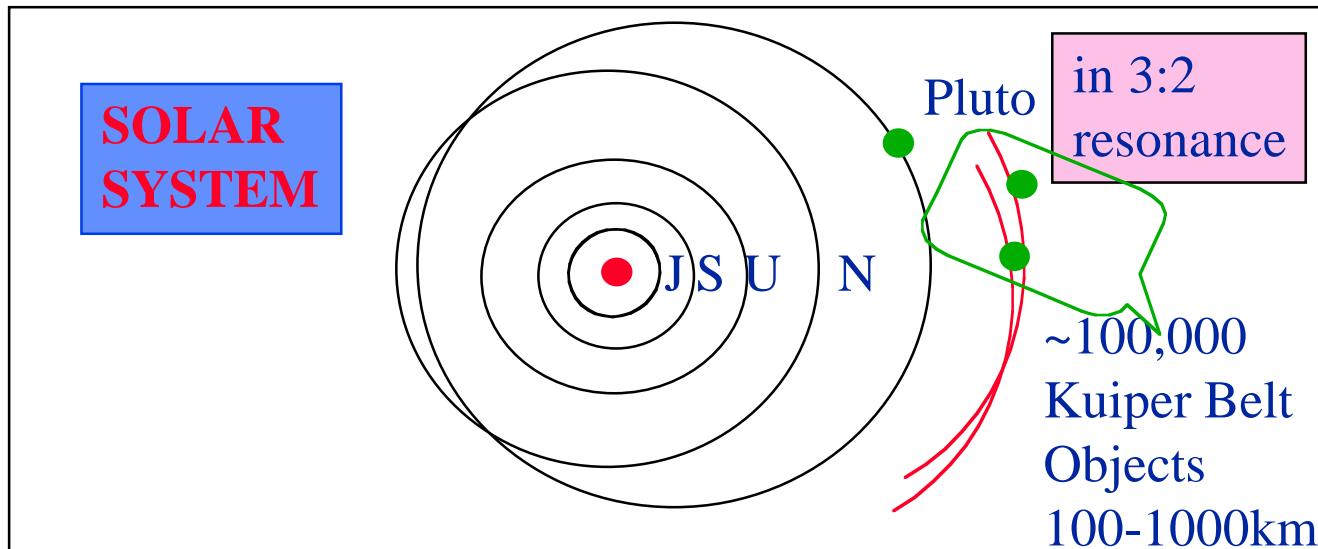
## SUMMARY

0

- (1) **New paradigm:** How to create and control a nontrivial V-state by starting from a trivial equilibrium and using a perturbation?
- (2) **Recent general solution (applicable to many nonlinear systems):** Pattern Formation by Synchronization (**PFS**) via passage through resonances.
- (3) We have seen emergence of m-fold symmetric **V-STATES** (nonlinear diocotron modes) by passage through resonances with a Kelvin mode.
- (4) Some nonuniform V-states are subject to 3-wave **decay instability**, but can be **stabilized** by negative feedback.
- (5) Some **30 papers in the field** in the last 10 years with applications in planetary dynamics (Plutinos), atomic physics, plasmas, fluids and nonlinear waves: [www.phys.huji.ac.il/~lazar](http://www.phys.huji.ac.il/~lazar)
- (6) **Many questions remain open:**  
different patterns and waves, other types of resonances, higher dimensionality, thresholds etc.

# THE PLUTINO PROBLEM

3

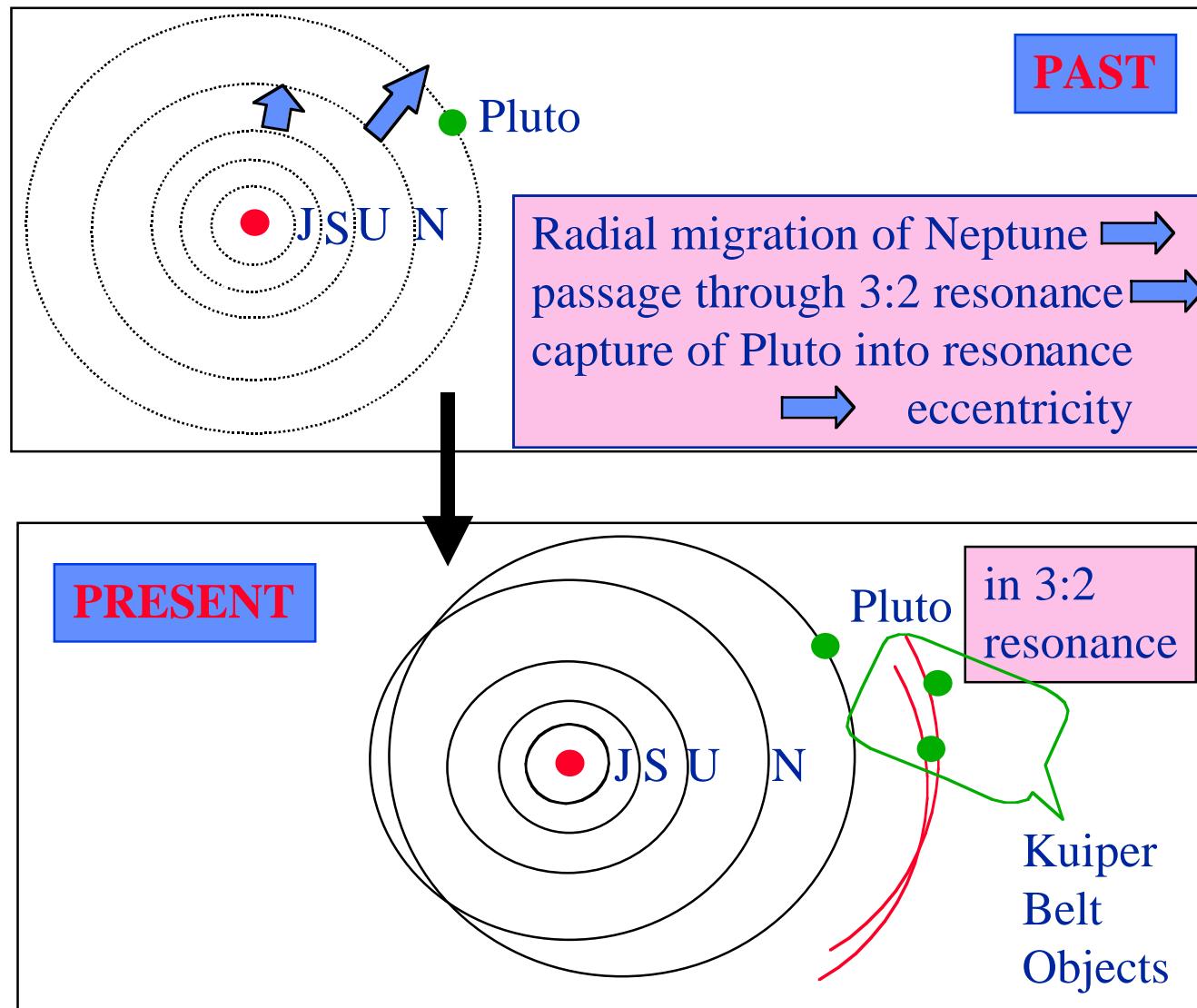


Distribution of KBOs (Jewitt, Lu)

- \*Why 30% of trans-Neptunian objects are in 3:2 resonance?
- \*Why they have large eccentricity?
- \*Why no objects in 2:1 resonance?

# MIGRATING PLANETS (Malhotra 1993-95)

2

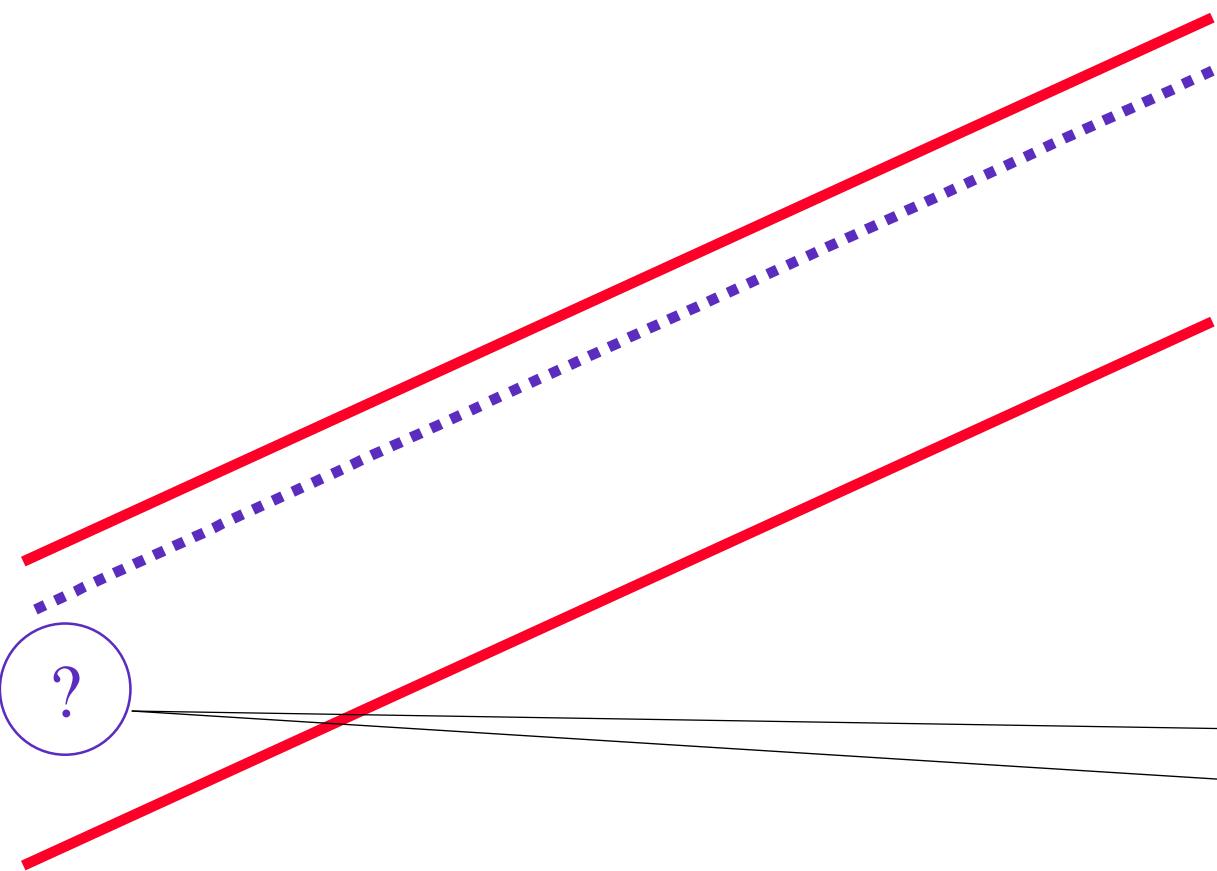


**BUT, why there are no KBOs in 2:1 resonance ???**

Inclusion of Sun's corotation creates  
an order of magnitude gap between capture  
thresholds into 2:1 and 3:2 resonances



$\alpha_{th}$



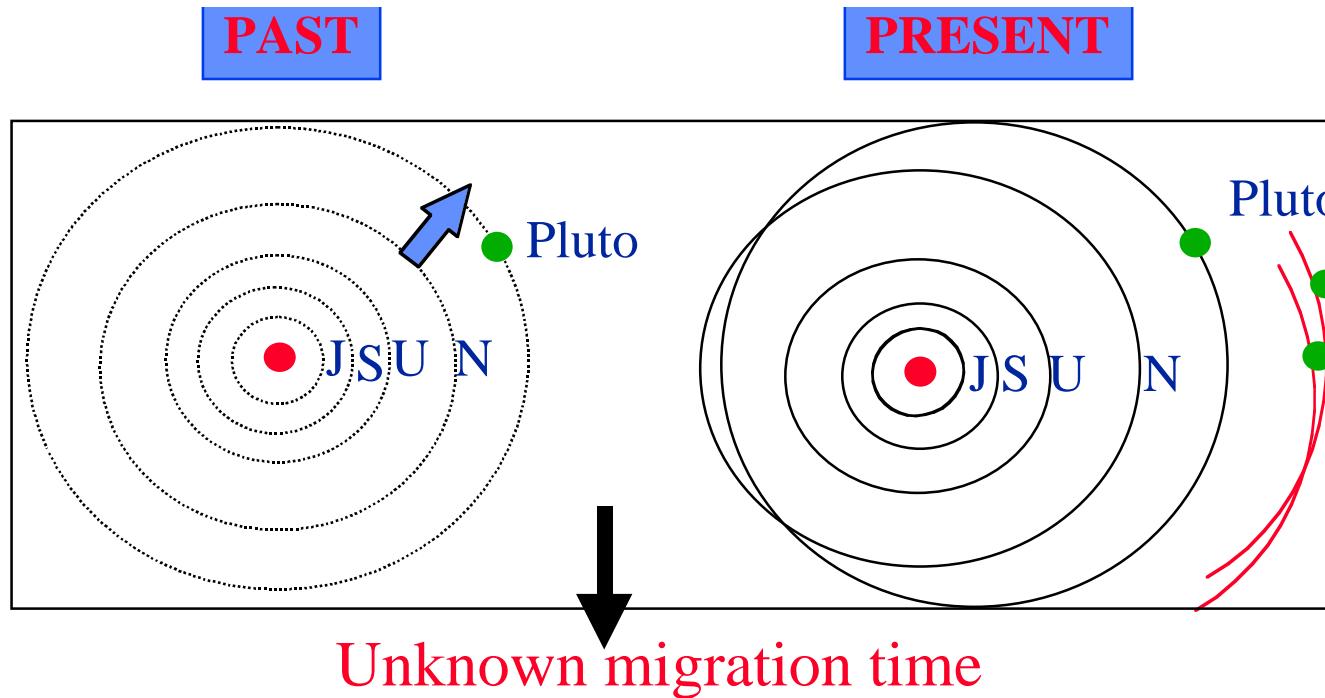
$\sim \varepsilon$

$$\begin{aligned} de/dt &= -\varepsilon \sin \Phi \\ d\Phi/dt &= \beta e^2 - \alpha t - \frac{\varepsilon}{e} \cos \Phi \\ \varepsilon &\sim m_2/m_1 \end{aligned}$$

$$\varepsilon_{th} \sim \alpha^{3/4}$$

$$\alpha_{th} \sim \varepsilon^{4/3}$$

Actual Neptune  
migration rate



Unknown migration time

Threshold phenomenon [Friedland, ApJ, 2001]:  
 Sun's motion increases the time scale  
 for capture into 2:1 resonance by factor of 10

$$2\text{Myr} = T_{\text{thr}}(3:2) < T_{\text{mig}} < T_{\text{thr}}(2:1) = 20\text{Myr}$$

**WE FOUND ACCURATE BOUNDS ON TIMESCALES INVOLVED IN  
 EARLY EVOLUTION OF THE SOLAR SYSTEM**